$((2/(\Omega\alpha))_{max} \approx 0.1)$ with a strong injection ($R \to -\infty$). Consequently, the locally uniform approximation is well satisfied for all injection strengths.

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NONLINEAR AZIMUTHAL WAVES IN A CENTRIFUGE

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Azimuthal wave motions in a liquid which partially fills a cylinder (centrifuge) rapidly rotating about a horizontal axis are discussed in this paper. Under the action of centrifugal force the liquid is pressed to the wall of the cylinder and moves together with it about the central air core. The vibrations of the free surface which arise are called centrifugal waves [1]. The difficulties of their theoretical investigation are related to the nonlinearity both of the basic equations and also of the boundary condition for the pressure on the free surface; therefore they have previously been studied only by linear methods [1, 2]. Nonlinear azimuthal waves in a centrifuge with an infinite radius of the rotating cylinder are analytically described below. The waves found are an analog of Gerstner trochoidal waves on a cylindrical surface. An approximate solution for a centrifuge with a finite outer radius is constructed by matching the waves obtained to the known linear ones.

1. We shall consider azimuthal waves in a centrifuge rotating at a constant angular velocity Ω. They have been investigated in the linear approximation in [2]. In the polar coordinate system R, θ rotating with velocity Ω the radial u and azimuthal v velocities are equal, respectively, to

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$$u(R,\theta,t)=\frac{\sigma_n d}{\left[(R_2/R_1)^{2n}-1\right]}\left(\frac{R}{R_1}\right)^{n-1}\left[\left(\frac{R_2}{R}\right)^{2n}-1\right]\sin(n\theta-\sigma_n t); \quad (1.1)$$

$$v(R,\theta,t)=\frac{\sigma_n d}{\left[(R_2/R_1)^{2n}-1\right]}\left(\frac{R}{R_1}\right)^{n-1}\left[\left(\frac{R_2}{R}\right)^{2n}+1\right]\cos(n\theta-\sigma_n t), \quad (1.2)$$

where R_1 and R_2 are the inner and outer radii of the unperturbed liquid ring, d is the amplitude of the sinusoidal profile on the free boundary, n is the azimuthal mode number, and σ_n is the frequency of the wave determined by the equality

$$\sigma_n^{\pm}=\frac{n\Omega}{\pm\left(n-1+(n+1)(R_2/R_1)^{2n}\right)^{1/2}\left((R_2/R_1)^{2n}-1\right)-1}. \quad (1.3)$$

Waves of frequency σ_n^+ move in the direction of rotation of the coordinate system, and those of frequency σ_n^- move in the opposite direction. In a fixed coordinate system both types of waves propagate in the direction of rotation of the flow. We note that the trajectories of the liquid particles are ellipses.

It is convenient to discuss waves of finite amplitude first for a centrifuge with an infinite outer radius R_2 (this corresponds to the case $R_2 \gg R_1$), for which it has proven possible to solve the problem exactly.

It has been shown in [3] that the system of equations of two-dimensional hydrodynamics is equivalent to the following equations:

$$\left(W_\eta(\overline{W})_{\bar\eta}-W_{\bar\eta}(\overline{W})_\eta\right)_t=0, \quad \left(W_{t\eta}(\overline{W})_{\bar\eta}-W_{t\bar\eta}(\overline{W})_\eta\right)_t=0. \quad (1.4)$$

Here $W = X + iY$, $\overline{W} = X - iY$, $\eta = a + ib$, $\bar\eta = a - ib$, X and Y are Eulerian coordinates, a and b are Lagrangian Cartesian coordinates, and t is the time; the subscripts denote differentiation with respect to the corresponding variable. Equations (1.4) express, respectively, the conditions of incompressibility of the liquid and conservation of vorticity along the trajectory.

One can convince oneself by direct substitution that the expression

$$W=G(\eta)e^{i\lambda t}+F(\bar\eta)e^{i\mu t}, \quad (1.5)$$

where λ and μ are real numbers and G and F are arbitrary analytic functions, is an exact solution of the system (1.4). It describes the class of rotational nonsteady flows of an ideal liquid, including the known exact solutions — Gerstner waves and the Kirchhoff elliptical vortex [4] — as particular cases. The trajectories of liquid particles for motions of the kind (1.5) will be epicycloids (hypocycloids), i.e., the particles describe a circle whose center in turn moves on a circle. Therefore the authors of [3] have proposed calling this type of flow Ptolemaic.

We shall assume that the waves being studied belong to the class of Ptolemaic motions; then the functions G and F are determined from the boundary conditions. Since at infinity the liquid rotates as a whole, one should set

$$G(\eta)=\eta,\ \lambda=\Omega \quad (1.6)$$

in the solution (1.5) and assume that $|F| \to 0$ as $|\eta| \to \infty$. We shall find the function F from the condition of constancy of the pressure on the free surface $|\eta| = R_1$. An expression is obtained for the pressure from the equations of motion in Lagrangian variables (for example, see [5]), and with the relationships (1.5) and (1.6) taken into account it takes the form

$$\frac{p}{\rho}=\frac{1}{2}\Omega^2|\eta|^2+\frac{1}{2}\mu^2|F|^2+\mathrm{Re}\int\left(\Omega^2\eta(\overline{F})_\eta+\mu^2\overline{F}\right)e^{i(\Omega-\mu)t}d\eta.$$

In order that the pressure on the free surface remain constant, vanishing of the coefficients of the temporal factors is necessary. This is satisfied if

$$F(\bar{\eta}) = A\bar{\eta}^{-q^2}, \quad q = \frac{\mu}{\Omega}, \tag{1.7}$$

where A is a constant.

We obtain the final expression for W by substituting Eqs. (1.6) and (1.7) into the solution (1.5):

$$W = \eta e^{i\Omega t} + A\bar{\eta}^{-q^2} e^{iq\Omega t}, \tag{1.8}$$

whence it is easy to conclude that the trajectories of the liquid particles are curtate epi- ($q > 0$) and hypocycloids ($q < 0$) with the number of cusps equal to $|q - 1|$, and the profiles of the propagating waves are epicycloids with the number of cusps equal to $q^2 - 1$; we shall assume q^2 to be an integer for time independence of the profile. Taking account of the form of the profile, we shall call the waves epicycloidal.

The pressure distribution in the liquid is specified by the formula

$$p = \frac{1}{2}\rho\Omega^2 q^2 A^2\left[|\eta|^{-2q^2} - R_1^{-2q^2}\right] + \frac{1}{2}\rho\Omega^2\left(|\eta|^2 - R_1^2\right), \tag{1.9}$$

i.e., the pressure is constant on the wave profile.

We shall also point out that for a constant A, which determines the amplitude of the waves, there is an upper limit $A = R_1^{q^2+1}/q^2$ when the profile of the free surface has cusps (for large values of A loops are formed on the profile — a physically unrealizable case).

Epicycloidal waves are rotational. The vorticity ω for them is written in the form

$$\omega = \frac{2\Omega\left(1 - q^5 A^2 |\eta|^{-2(q^2+1)}\right)}{1 - q^4 A^2 |\eta|^{-2(q^2+1)}},$$

whence it is evident that the vorticity will be an alternating function for waves with positive values of q.

We shall find the angular rotational velocity of the waves Ω_0. It is evident that the rotation of the liquid as a whole with frequency Ω_0 is characterized by the common factor $\exp(i\Omega_0 t)$ in the expression for W; therefore the solution (1.9) will take the form

$$W = \eta e^{i(\Omega-\Omega_0)t} + A\bar{\eta}^{-q^2} e^{i(q\Omega-\Omega_0)t}$$

in the reference frame in which the profile is motionless. In this system the trajectories of the liquid particles coincide with the shape of the profile; therefore the equality $q\Omega - \Omega_0 = q^2(\Omega - \Omega_0)$ is valid, from which we find $\Omega_0 = q(q + 1)^{-1}\Omega$.

In the reference frame moving with angular velocity Ω the frequency of rotation of the wave profile is equal to $(q + 1)^{-1}\Omega$, so that waves corresponding to negative q move in the direction of rotation and waves corresponding to positive q move in the opposite direction. The rotational velocity of the profile for linear waves in this same reference system is equal to $\sigma_n^{\pm}/n$, and in the case of an infinite radius of the centrifuge ($R_2 \to \infty$) it is written as

$$\frac{\sigma_n^{\pm}}{n} = -\frac{\Omega}{(\mp\sqrt{n+1}+1)}.$$

Setting $n = q^2 - 1$, we obtain that the frequencies of rotation of the profiles of linear and epicycloidal waves coincide (the plus sign in front of the root pertains to waves with $q > 0$, and the minus sign corresponds to waves with $q < 0$). We shall use this property of epicycloidal waves to find an approximate solution for waves in a centrifuge with a finite wall radius.

2. It is evident that the exact solution for epicycloidal waves becomes invalid for a centrifuge with a finite outer radius, since the normal component of the velocity on the wall does not vanish. At the same time linear waves (1.1)-(1.3) satisfy this condition; therefore we shall assume that nonlinear rotational waves localized near the free surface are matched at some depth to linear ones. It is clear that for such wave motion the boundary

conditions on the free surface and the wall will be satisfied and it remains only to satisfy the continuity conditions of the normal velocity and the pressure on the splice boundary.

The matching surface $R = R^*$ emerges for linear waves as a free surface (one should replace R_1 by R^* in formulas (1.1)-(1.3)). We shall assume that the amplitude of the deviation of the profile on this surface d is small: $d \ll n^{-1} (R_1 - R^*)$ (we note that the amplitude of vibrations of the free surface may be of the order of the wavelength); then the radial velocities of the waves will be approximately normal to the splice boundary. In a reference system rotating with angular velocity Ω the radial velocity u^* of the exact solution (1.9) is of the form

$$u^* = -(q-1)\Omega A r^{-q^2} \sin[(q^2-1)\varphi + (q-1)\Omega t], \quad (2.1)$$

where r and φ are the modulus and phase, respectively, of the complex Lagrangian coordinate η. Due to the smallness of d, we shall replace the Eulerian coordinates by Lagrangian ones in formula (1.1); then

$$u = d\sigma_n \sin(n\varphi - \sigma_n t). \quad (2.2)$$

We shall consider waves for which the inequality $(R_2/R^*)^n \gg 1$ is valid; in this case $\sigma_n = -(q-1)\Omega$ (we recall that $n = q^2 - 1$), and comparing relations (2.1) and (2.2), we obtain

$$R^* = (A/d)^{1/q^2}.$$

In the fixed coordinate system the pressure on the surface $r = |\eta| = R^*$, which is determined by formula (1.9), is constant and emerges for linear waves (1.1)-(1.3) in the role of the pressure on the free surface; therefore the continuity condition is easily satisfied by the selection of the appropriate constant in the expression for the pressure of linear waves [2].

We note that in the rotating reference frame liquid particles move on circular trajectories in epicycloidal waves and on ellipses, which degenerate on the wall into straight lines, in linear waves.

We shall also point out that the effects of surface tension, which have been neglected in this paper, are unimportant up to wavelengths of the order of $(T/\rho\Omega^2 R_1)^{1/2}$, where T is the surface tension coefficient and ρ is the density; therefore there exists a lower limit to the wavelength of the approximate solution obtained: $R_1/(q^2 - 1) \gg (T/(\rho\Omega^2 R_1))^{1/2}$.

3. The properties of epicycloidal waves are very similar to the properties of Gerstner trochoidal waves on the surface of an infinitely deep liquid. For both types of waves the pressure on the wave profile is constant; in the reference frame in which the waves are motionless the trajectories of the liquid particles are circles. Finally, the form of the free surface of these waves is determined by the related curves — an epicycloid and a trochoid, so that in essence these are Gertsner waves on a cylindrical surface. At the same time it should be noted that in contrast to Gerstner waves the actual source of the vorticity for epicycloidal waves is clear — the rotating liquid.

Continuing to draw analogies between the two types of wave motions, we shall point out that it is easy to obtain for waves in a heavy liquid of finite depth H an approximate solution similar to that discussed in Sec. 2 in which Gertsner waves are matched to linear gravity waves at some depth H^*, whose value is determined from the condition of continuity of the normal component of the velocity. It is necessary to impose the following constraints on the wavenumber k: $kB \exp(-kH^*) \ll 1$ (smallness of the wave amplitude on the matching boundary in comparison with the wavelength); here B is the rise amplitude of the free surface; in the general case it is not small and has the upper limit $1/k$; $k(H - H^*) \gg 1$ (absence of a tangential discontinuity on the matching boundary) and $k(T/\rho g)^{1/2} \ll 1$ (neglect of capillarity, and g is the free-fall acceleration).

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BOILING MODEL FOR A FLUIDIZED BED OF PARTICLES

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A mechanism for boiling of a fluidized bed was examined in [1]. Due to hydrodynamic instability the solid particles acquire random motion, and as a result of collisions between particles part of the energy of random motion is converted to rotation of the particles. A rotating particle experiences a Magnus force which considerably increases the random motion and leads to spontaneous boiling of the layer. For this mechanism there is typically a minimum boiling time τ_γ, defined basically as the time to develop a hydrodynamic instability. It is shown in this study that besides the spontaneous mechanism there is an induced mechanism for boiling of the bed arising from the generation of random motion in one particle layer. Particles in that layer boil, transmitting a perturbation to the energy of the next layer, and leading to layer boiling in a manner analogous to the propagation of a detonation wave in solids.

1. We consider a bed of rather densely packed spherical particles at rest, supported on a grating permeable to gas, through which gas is circulated from below. When a certain gas velocity is reached the particles become "weightless", i.e., the gravity force becomes equal to the drag force. Such a bed of gas and particles is conventionally called fluidized. However, this state of the bed is unstable, and after a certain time the bed boils. The behavior of the particles in a boiling bed is reminiscent of that of gas molecules, and therefore by analogy we shall call them a gas of particles. The system of equations describing the motion of the mixture, allowing for the Magnus force, as given in [2], and in the notation adopted in [3], has the form

$$\begin{gathered}
\frac{\partial\rho_1}{\partial t}+\nabla(\rho_1\mathbf{v}_1)=0,\quad \rho_1=\rho_{11}m_1,\\
\frac{\partial\rho_2}{\partial t}+\nabla(\rho_2\mathbf{v}_2)=0,\quad \rho_2=\rho_{22}m_2,\\
\rho_1\frac{d_1\mathbf{v}_1}{dt}=-m_1\nabla p_1-\mathbf{f}_{12},\quad \frac{d_1(\ )}{dt}=\frac{\partial(\ )}{\partial t}+(\mathbf{v}_1\cdot\nabla),\\
\rho_2\frac{d_2\mathbf{v}_2}{dt}=-m_2\nabla p_1+\mathbf{f}_{12}-\nabla p_2+\mathbf{g}\rho_2,\\
\frac{d_1e_1}{dt}+p_1\frac{d_1}{dt}\left(\frac{1}{\rho_{11}}\right)=\mathbf{f}_{12}(\mathbf{v}_1-\mathbf{v}_2)\Big/\rho_1-\frac{q_{12}}{\rho_1}-\frac{Q_M}{\rho_1},\\
\frac{d_2e_2}{dt}=\frac{\dot q_{12}}{\rho_2}+\frac{Q_D}{\rho_2},\quad \dot q_{12}=\varkappa(T_1-T_2),\quad e_2=c_2T_2,\\
\frac{d_2(\ )}{dt}=\frac{d(\ )}{\partial t}+(\mathbf{v}_2\cdot\nabla),\quad e_1=c_1T_1,\quad \varepsilon_2=3c^2,\\
\frac{d_2\varepsilon_2}{dt}=-p_2\frac{d_2}{dt}\left(\frac{1}{\rho_2}\right)+\frac{\dot Q}{\rho_2},\quad \psi=1-1{,}17m_2^{2/3},
\end{gathered}\tag{1.1}$$

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